# Lecture 9 - further remarks on QCoh

Tuesday, October 04, 2016 3:28 AM

## Returning to $QCoh(\mathcal{X})$

**Lem:** For a geometric stack, $QCoh(\mathcal{X})$ has enough coherent sheaves, meaning every $F$ is union of coherent subsheaves

even more!

$QCoh(\mathcal{X}) = Ind(Coh(\mathcal{X}))$, meaning $Coh(\mathcal{X})$ are fin. pres. objects, and

$$QCoh(\mathcal{X}) \longrightarrow Fun(Coh(\mathcal{X})^{op}, Ab)$$

is an equivalence

i.e. every QCoh is filtered colimit of coherent sheaves

**Eg** $\mathcal{X} = Spec(A)/G$

$\Longrightarrow QCoh(\mathcal{X}) \cong$ G-equiv. A-mod.

**Thm:** For an alg. stack, $QCoh(\mathcal{X})$ is a Grothendieck abelian category, meaning

**Tag 079A**

From <http://stacks.math.columbia.edu/tag/079A>

**Definition 19.10.1.** Let $\mathcal{A}$ be an abelian category. We name some conditions

| | |
|---|---|
| AB3 | $\mathcal{A}$ has direct sums, |
| AB4 | $\mathcal{A}$ has AB3 and direct sums are exact, |
| AB5 | $\mathcal{A}$ has AB3 and filtered colimits are exact. |

Here are the dual notions

| | |
|---|---|
| AB3* | $\mathcal{A}$ has products, |
| AB4* | $\mathcal{A}$ has AB3* and products are exact, |
| AB5* | $\mathcal{A}$ has AB3* and filtered limits are exact. |

We say an object $U$ of $\mathcal{A}$ is a *generator* if for every $N \subset M$, $N \neq M$ in $\mathcal{A}$ there exists a morphism $U \to M$ which does not factor through $N$. We say $\mathcal{A}$ is a *Grothendieck abelian category* if it has AB5 and a generator.

<u>Thm:</u> In a Grothendieck category, $\exists$ enough injectives <u>and</u> enough "K-injective" complexes

$\Longrightarrow$ can construct the unbounded derived category of quasi-coherent sheaves

→ Localization of homotopy category of complexes by class of q-iso.'s

→ <u>or</u> the homotopy category of K-injective complexes!

(See Spaltenstein)

<u>Rem:</u> There are other constructions of the derived category which will agree with this in case of affine [illegible]

with this in case of affine $\mathcal{U}$
diagonal.

<u>Functors</u>: For any map of stacks
$f: \mathcal{X} \to \mathcal{Y}$, get a pullback
$f^*: QCoh(\mathcal{Y}) \longrightarrow QCoh(\mathcal{X})$

$\hookrightarrow$ can construct a pushforward $f_*$ which is right-adjoint by Freyd's adjoint functor thm. $Rf_*$ is the derived functor

<u>Eg</u>: $\mathcal{X} = X/G$, $F \in QCoh(\mathcal{X})$, then

$\Gamma(\mathcal{X}, F)$ factors as composition

$$X/G \xrightarrow[p]{} \cdot/G \longrightarrow \bullet$$

First is $R\Gamma(X, F)$, next is $(\cdot)^G$

$\rightsquigarrow$ if $G$ is lin. reductive, no need to derive the invariants functor

<u>Remark</u>: For a geometric stack, can compute cohomology with a Čech resolution

$\hookrightarrow$ $U_0 \xrightarrow{\text{aff.}} \mathcal{X}$, construct Čech nerve $U_\bullet$

$F^\bullet$ a complex in $Ch(QCoh(\mathfrak{X}))$, can restrict to $F_n^\bullet \in Ch(QCoh(U_n))$

↳ All $U_n$ affine so can regard $F_n^\bullet$ as a complex of abelian groups, then there is a double complex

$$F_0^\bullet \to F_1^\bullet \to F_2^\bullet \cdots$$

And $\Gamma(\mathfrak{X}, F^\bullet) \cong \mathrm{Tot}^{prod}(F_0^\bullet \to F_1^\bullet \to \cdots)$

<u>Exc:</u> compute $R\Gamma({}^{\bullet}/_{(\mathbb{Z}/p)}, \mathcal{O})$ over a field in every characteristic.